Int. J. Appl. Math. Comput. Sci., 2024, Vol. 34, No. 1, 15–27
DOI: 10.61822/amcs-2024-0002

THE EXISTENCE OF MILD SOLUTIONS AND APPROXIMATE CONTROLLABILITY FOR NONLINEAR FRACTIONAL NEUTRAL EVOLUTION SYSTEMS

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The existence of mild solutions and approximate controllability for Riemann–Liouville fractional neutral evolution systems with nonlocal conditions of a fractional order is investigated. The Laplace transform and semigroup theory are the tools used to prove the existence. In turn, approximate controllability is proved on the basis of a Nemytskii operator, a Mittag-Leffler function and certain hypotheses using fixed point theorems, as well as the construction of a Cauchy sequence. An example is provided to highlight the main results.

Keywords: fractional neutral evolution system, approximate controllablity, mild solution, Riemann–Liouville derivatives.

1. Introduction

Fractional differential equations are a generalization of classical ones, and they have been applied to model a variety of biological, physical, mechanical, and engineering problems (Hu *et al.*, 2009; Zhang, 2006; Podlubny, 1999; Diethelm and Freed, 1999; Mainardi, 1997; Zhou and Jiao, 2010b). A wide range of practical applications has prompted the publication of several existing results incorporating the two classical derivatives Caputo and Riemann–Liouville for systems of order $0 < q < 1$ (Kiryakova 1994; Samko *et al.*, 1993; Diethelm and Ford, 2004; Ech-chaffani *et al.*, n.d.; 2022) and for order $1 < q < 2$ (Li *et al.*, 2016; 2013; Zhou and He, 2021; Li, 2015).

Neutral differential equations are those where the time delay appears under the derivative of the unknown functions; in fact, many control systems are governed by neutral differential equations. That is why dealing with such systems is more complex than with the classical ones, where the delays only occur in the state. This type of delays can be complex to handle, but it improves the performance of the system in which they occur. Adding

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neutral delays to a fractional system is more beneficial as such a system does have memory. Neutral systems can simulate a wide range of natural phenomena from many fields such as fluid dynamics, electronics, biological models, chemical dynamics, etc. (see Chang and Liu, 2009; Mokkedem and Fu, 2014). Classical differential equations cannot describe most of these phenomena, such as heat conduction in a fading memory material, and anomalous diffusion.

Controllability is a very important concept in contemporary science and technology. Control is ubiquitous in our daily lives; it exists in our cars as we drive and maintain the car on the road, in the braking system, and so on. Broadly speaking, the concept of exact controllability of a system means transferring the system from an initial state to a final state within a fixed time interval, although this is sometimes difficult to achieve; there is a weaker concept, that of approximate controllability, where systems have a weaker conceptualization and a wider application. In general, it is difficult to achieve exact controllability for differential systems in infinite-dimensional Banach spaces, and many diffusion control systems are not exactly controllable since the corresponding linear operator generates a compact semi group, for example, the heat equation. We refer the readers to the works of Mahmudov (2003), Triggiani (1977), Xi *et al.* (2022b) or Bárcenas *et al.* (2005). As a consequence, many researchers are devoted to the study of approximate controllability because it is more realistic and adequate in many real situations (Li *et al.*, 2021; Zhao and Liu, 2022). Nevertheless, the theory of approximate controllability for fractional equations is still in its early stages of development.

Liu and Li (2015) proved the approximate controllability of the following fractional evolution control system involving a Riemann–Liouville fractional derivative of order $0 < q \le 1$, by transforming the control problem into a fixed point one:

$$\begin{cases} D_t^q x(t) = \mathcal{A}x(t) + \mathcal{B}u(t) + f(t, x(t)), \\ \qquad t \in (0,T], \\ I_t^{1-q} x(t)\Big|_{t=0} = x_0 \in X. \end{cases}$$

In recent years, there has been a significant interest in the controllability of fractional neutral differential systems with a Caputo derivative of order $0 < q < 1$. Sakthivel *et al.* (2012) examined exact controllability under some assumptions of the abstract fractional neutral evolution control system

$$\begin{cases} {}_0^C D_t^q [x(t) - h(t, x_t)] = \mathcal{A}x(t) + \mathcal{B}u(t) + f(t, x_t), \\ \qquad t \in J = [0,T], \\ x_0(\theta) = \phi(\theta), \quad \theta \in [-r, 0]. \end{cases}$$

Vijayakumar *et al.* (2021) studied the approximate controllability of neutral integro-differential system inclusions of a Sobolev type with infinite delay. For the case of hyperbolic neutral systems, Xi *et al.* (2022a) investigated the existence and approximate controllability of a fractional neutral evolution system of the following form

$$\begin{cases} {}_0^C D_t^q [x(t) - g(t, x_t)] = \mathcal{A}x(t) + \mathcal{B}u(t) + f(t, x_t), \\ \qquad t \in (0,T], \\ x(\theta) = \phi(\theta), \quad \theta \in [-r, 0], \\ x'(0) = a, \end{cases}$$

where ${}_0^C D_t^q$ signifies the Caputo fractional derivative of order $1 < q < 2$. The controllability problem in all the above works has been discussed using Shauder's fixed point theorem. For more applications on several dynamics, we recommend the work of (Babiarz *et al.*, 2016). Other researchers have been interested in studying the approximate controllability of fractional neutral systems with non-local conditions, with state-dependent delays and for impulsive systems. For more details, we refer the readers to the works of Dhayal *et al.* (2019), Mingyuan *et al.* (2016), Yan (2012), Du *et al.* (2020), Zhu *et al.* (2020), Agarwal *et al.* (2022), Leiva and Sundar (2017) or Liang (2022) and the references therein. The study of the controllability for various integer or fractional Caputo derivatives of certain linear and non linear systems seems to have been completed. However, the approximate controllability of fractional neutral Riemann–Liouville systems with an analytic semigroup is still open.

Inspired by the above studies and related work, in this paper we consider fractional neutral systems, with a Riemann–Liouville derivative of order $0 < q < 1$ given by

$$\begin{cases} {}_0^{RL} D_t^q [z(t) - h(t, z(t))] \\ \qquad = -\mathcal{A}z(t) + \mathcal{B}u(t) + f(t, z(t)), \\ \qquad t \in J = (0,T], \quad T \ge 1, \\ I_t^{1-q}[z(t) - h(t,)]|_{t=0} = z_0, \end{cases} \tag{1}$$

where ${}_0^{RL} D_t^q$ is the fractional derivative of order $0 < q \le 1$ (see Definition 1), $-\mathcal{A} : D(\mathcal{A}) \subseteq Z \to Z$ is the infinitesimal generator of an analytic C_0 semigroup $\{\mathcal{T}(t)\}_{t\ge 0}$ on a Banach space $(Z, \|\cdot\|)$. The control function u belongs to the space $\mathcal{U} = L^p([0,T]; U)$, $p > 1/q \ge 1$, where U is a Banach space. $\mathcal{B}$ is a linear operator defined from $\mathcal{U}$ onto $L^p([0,T]; Z)$, z_0 is the initial state, $z(t)$ is the state that depends on time t, and $h, f : [0,T] \times Z^q \to Z$ are two given functions.

We prove the existence of a mild solution of the system given by (1) semigroup using the Laplace transformation combined with semigroup properties. Furthermore, we aim to prove the approximate controllability of such a system under Assumptions H_1–H_4; see Section 4. A strong motivation for studying the initial conditions of Riemann–Liouville integrals comes from the fact that there are many physical problems that can be modeled efficiently using Riemann–Liouville integrals and cannot be modeled using classical initial value problems, an example being the fractional Maxwell model describing a realistic behavior for a viscous solid in which a stress relaxation at $t = 0$ is modeled by a fractional derivative (Heymans and Podlubny, 2006). For more applications, a detailed review is presented by Diethelm and Ford (2004).

The key aspects of the present work are stated below:

(i) The system (1) is of a nonlinear evolution neutral type that is studied in Banach spaces.

(ii) Under various settings, sufficient conditions for the existence and uniqueness of the mild solution are given.

(iii) The approximate controllability of the system (1) is presented. A fractional neutral partial differential system is used as an example to illustrate the validity of our main results.

(iv) A complexity arises from the existence of a delay in the derivative for neutral systems and the initial condition of the system, when proving sufficient conditions for the existence of the solution and approximate controllability, since fractional Riemann–Liouville derivatives have a singularity at zero and fractional differential equations in the sense of Riemann–Liouville require initial conditions of a special form lacking physical interpretation (Kilbas *et al*., 2006).

The rest of the paper is structured as follows. In Section 2, we provide essential preliminaries. In Section 3, through the Laplace transform and semigroup properties, a formula for the mild solution of the system (1) is provided, and in Theorem 1, we prove its existence and uniqueness. In Section 4, we focus on the weak controllability of the system (1) and prove our second main result stated in theorem 2. Finally, an example is provided to illustrate the feasibility of our main results in Section 5.

2. Preliminaries

We recall some essentials of semigroup theory and fractional derivatives, which can be found in the works of Podlubny (1999), Pazy (1983), Kilbas *et al*. (2006) and the references therein.

Let $z : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function.

Definition 1. (*Podlubny, 1999*)

(i) Let $q > 0$; the Riemann–Liouville integral of the function z of order q is defined by

$$I^q z(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} z(s)\,\mathrm{d}s, \qquad t > 0.$$

(ii) The fractional derivative of order q of z in the sense of Riemann–Liouville is defined by

$$\begin{aligned} &{}_0^{RL}D_t^q z(t) \\ &\quad = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-q-1} z(s)\,\mathrm{d}s, \\ &\qquad\qquad t > 0, \end{aligned}$$

where $q \in]n-1, n[$, $n \in \mathbb{N}$.

Let

$$\begin{aligned} &AC^m(J,Z) \\ &\quad = \left\{ f : J \to Z \text{ and } f^{(m-1)}(z) \in AC(J,Z) \right\}, \end{aligned}$$

where $AC(J,Z)$ is the set of absolutely continuous functions from J onto Z endowed with the norm $\|z\|_\infty = \sup_{t\in J} |z(t)|$.

Lemma 1. (Kilbas *et al.*, 2006) *Let $q > 0$, $m = [q] + 1$, and let $z_{m-q}(t) = I_t^{m-q} z(t)$ be the fractional integral of order $m - q$. If $z \in L^1(J,Z)$ and $z_{m-q} \in AC^m(J,Z)$, then we have*

$$I_t^q {}_0^{RL}D_t^q z(t) = z(t) - \sum_{k=1}^m \frac{z_{m-q}^{(m-k)}(0)}{\Gamma(q-k+1)} t^{q-1}.$$

The Laplace transform for a Riemann–Liouville fractional integral is given by

$$L\left\{ I_t^q z(t); \gamma \right\} = \frac{1}{\gamma^q} \nu(\gamma),$$

where $\nu(\gamma)$ is the Laplace transform of z.

Consider the Banach space

$$C_{1-q}(J,Z) = \left\{ z : t^{1-q} z(t) \in C(J,Z),\ 0 < q \le 1 \right\},$$

equipped with the norm

$$\|z\|_{C_{1-q}} = \sup \left\{ t^{1-q} \|z(t)\| : t \in J,\ 0 < q \le 1 \right\}.$$

Let us suppose that $-\mathcal{A}$ generates a compact and uniformly bounded C_0 semigroup $\{\mathcal{T}(t)\}_{t\ge 0}$; if $0 \in \rho(\mathcal{A})$ (resolvent of $\mathcal{A}$), then we define the fractional power $\mathcal{A}^\lambda$ as a closed linear operator on its domain $D\left(\mathcal{A}^\lambda\right)$, for all $0 < \lambda \le 1$. We state the following assertions which we will use to justify our findings (see Pazy, 1983):

(i) There is $M \ge 1$ such that

$$M = \sup_{t \ge 0} |\mathcal{T}(t)| < \infty. \tag{2}$$

(ii) For any $\lambda \in]0,1]$, there exists $C_\lambda > 0$ such that

$$|\mathcal{A}^\lambda \mathcal{T}(t)| \le \frac{C_\lambda}{t^\lambda}, \quad t \in]0,T]. \tag{3}$$

3. Mild solution for a neutral system

In this section, our main purpose is to give sufficient conditions for the existence and uniqueness of the mild solution to the neutral system (1).

3.1. Formula for the mild solution.

The following lemma is a motivation to define the mild solution for the system (1).

Lemma 2. *Under Assumptions H_1–H_4, let $q \in]0,1]$ with $p > 1/q \geq 1$. If $z(t) \in L^1(J,Z)$, $z_{1-q} \in AC(J,Z))$, and z is a solution of the system*

$$\begin{cases} {}_0^{RL}D_t^q\left[z(t) - h\left(t, z(t)\right)\right] = -\mathcal{A}z(t) + f(t, z(t)), \\ \qquad\qquad t \in J = (0,T], \quad T \geq 1, \\ I_t^{1-q}[z(t) - h(t,z(t))]|_{t=0} = z_0, \end{cases} \tag{4}$$

then z satisfies the following integral equation:

$$\begin{aligned} z(t) &= t^{q-1}\mathcal{T}_q(t)z_0 + h(t,z(t)) \\ &\quad - \int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{T}_q(t-s)h(s,z(s))\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f(s,z(s))\,\mathrm{d}s, \end{aligned}$$

where

$$\mathcal{T}_q(t)z = q\int_0^\infty \theta\Psi_q(\theta)\mathcal{T}(t^q\theta)z\,\mathrm{d}\theta$$

and

$$\Psi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\Phi_q\Big(\theta^{-\frac{1}{q}}\Big),$$

$$\Phi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),$$

$\theta \in (0,\infty)$.

Proof. Applying the Riemann–Liouville integral operator to (4) when combined with Lemma 1 yields

$$\begin{aligned} z(t) &= h(t,z(t)) + \frac{I_t^{1-q}\left[z(t) - h(t,z(t))\right]|_{t=0}}{\Gamma(q)}t^{1-q} \\ &\quad - I_t^q\mathcal{A}z(t) + I_t^q f(t,z(t)) \\ &= h(t,z(t)) + \frac{t^{q-1}}{\Gamma(q)}z_0 \\ &\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[-\mathcal{A}z(s) + f(s,z(s))\big]\,\mathrm{d}s. \end{aligned} \tag{5}$$

Let $\gamma > 0$. Then, if we plug the following Laplace transforms:

$$\nu(\gamma) = \int_0^\infty e^{-\gamma s}z(s)\,\mathrm{d}s,$$

$$\mu(\gamma) = \int_0^\infty e^{-\gamma s}f(s,z(s))\,\mathrm{d}s,$$

$$\xi(\gamma) = \int_0^\infty e^{-\gamma s}h(s,z(s))\,\mathrm{d}s.$$

into (5), we obtain

$$\begin{aligned} \nu(\gamma) &= \frac{1}{\gamma^q}z_0 + \xi(\gamma) - \frac{1}{\gamma^q}\mathcal{A}v(\gamma) + \frac{1}{\gamma^q}\mu(\gamma) \\ &= (\gamma^q I + \mathcal{A})^{-1}z_0 + \gamma^q(\gamma^q I + \mathcal{A})^{-1}\xi(\gamma) \\ &\quad + (\gamma^q I + \mathcal{A})^{-1}\mu(\gamma) \\ &= \int_0^\infty e^{-\gamma^q s}\mathcal{T}(s)z_0\,\mathrm{d}s \\ &\quad + \gamma^q\int_0^\infty e^{-\gamma^q s}\mathcal{T}(s)\xi(\gamma)\,\mathrm{d}s \\ &\quad + \int_0^\infty e^{-\gamma s}\mathcal{T}(s)\mu(\gamma)\,\mathrm{d}s, \end{aligned} \tag{6}$$

where I is the operator identity on Z. Let

$$\Phi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),$$

$\theta \in (0,\infty)$, and its Laplace transform be given by

$$\int_0^\infty e^{-\gamma\theta}\Phi_q(\theta)\,\mathrm{d}\theta = e^{-\gamma^q}, \quad q \in (0,1). \tag{7}$$

Hence, from (5) and (7) it follows that

$$\begin{aligned} &\int_0^\infty e^{-\gamma^q s}\mathcal{T}(s)z_0\,\mathrm{d}s \\ &= q\int_0^\infty s^{q-1}e^{-(\gamma s)^q}\mathcal{T}(s^q)z_0\,\mathrm{d}s \\ &= q\int_0^\infty\int_0^\infty s^{q-1}e^{-\gamma s\theta}\Phi_q(\theta)\mathcal{T}(s^q)z_0 d\theta\,\mathrm{d}s \\ &= q\int_0^\infty\int_0^\infty e^{-\gamma s}\Phi_q(\theta)\mathcal{T}\left(\frac{s^q}{\theta^q}\right)\frac{s^{q-1}}{\theta^q}d\theta z_0\,\mathrm{d}s \\ &= \int_0^\infty e^{-\gamma s}\left[q\int_0^\infty\Phi_q(\theta)\mathcal{T}\left(\frac{s^q}{\theta^q}\right)\frac{s^{q-1}}{\theta^q}d\theta\right]z_0\,\mathrm{d}s, \end{aligned} \tag{8}$$

$$\begin{aligned} &\gamma^q\int_0^\infty e^{-\gamma^q s}\mathcal{T}(s)\xi(\gamma)\,\mathrm{d}s \\ &= \int_0^\infty\int_0^\infty q\gamma^q t^{q-1}e^{-(\gamma s)^q}\mathcal{T}(s^q)e^{-\gamma t}h(t,z(t))\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_0^\infty\frac{\mathrm{d}}{\mathrm{d}s}e^{-(\gamma s)^q}\left[\int_0^\infty -\mathcal{T}(s^q)e^{-\gamma t}h(t,z(t))\,\mathrm{d}t\right]\,\mathrm{d}s \\ &= \left(e^{-(\gamma s)^q}\int_0^\infty -\mathcal{T}(s^q)e^{-\gamma s}h(s,z(s))\,\mathrm{d}s\right)\Big|_{s=0}^{s=\infty} \\ &\quad + \int_0^\infty\int_0^\infty qs^{q-1}e^{-(\gamma s)^q}\mathcal{A}\mathcal{T}(s^q)e^{-\gamma t}h(t,z(t))\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_0^\infty e^{-\gamma s}\Big[h(s,z(s)) \\ &\quad + q\int_0^s\int_0^\infty\Phi_q(\theta)\mathcal{A}\mathcal{T}\left(\frac{(s-t)^q}{\theta^q}\right) \\ &\quad \times h(s,z_s)\frac{(s-t)^q}{\theta^q}\,\mathrm{d}\theta\,\mathrm{d}t\Big]\,\mathrm{d}s, \end{aligned} \tag{9}$$

$$\begin{aligned}
&\int_0^\infty e^{-\gamma^q s}\mathcal{T}(s)\mu(\gamma)\,\mathrm{d}s\\
&=\int_0^\infty\int_0^\infty qs^{q-1}e^{-(\gamma s)^q}\mathcal{T}(s^q)e^{-\gamma t}f(t,z(t))\,\mathrm{d}t\,\mathrm{d}s\\
&=\int_0^\infty\int_0^\infty\int_0^\infty q\Phi_q(\theta)e^{-\gamma s\theta}\mathcal{T}(s^q)e^{-\gamma t}s^{q-1}\\
&\quad\times f(t,z(t))\,\mathrm{d}\theta\,\mathrm{d}t\,\mathrm{d}s\\
&=\int_0^\infty\int_0^\infty\int_0^\infty q\Phi_q(\theta)e^{-\gamma s}\mathcal{T}\left(\frac{s^q}{\theta^q}\right)\frac{s^{q-1}}{\theta^q}e^{-\gamma t}\\
&\quad\times f(t,z(t))\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}t\\
&=\int_0^\infty\int_0^\infty\int_0^\infty q\Phi_q(\theta)e^{-\gamma(s+t)}\mathcal{T}\left(\frac{s^q}{\theta^q}\right)\frac{s^{q-1}}{\theta^q}\\
&\quad\times f(t,z(t))\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}t\\
&=\int_0^\infty\int_0^\infty\int_t^\infty q\Phi_q(\theta)e^{-\gamma s}\mathcal{T}\left(\frac{(s-t)^q}{\theta^q}\right)\frac{(s-t)^q}{\theta^q}\\
&\quad\times f(t,z(t))\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}t\\
&=\int_0^\infty e^{-\gamma s}\Big[q\int_0^s\int_0^\infty\Phi_q(\theta)\mathcal{T}\left(\frac{(s-t)^q}{\theta^q}\right)\frac{(s-t)^q}{\theta^q}\\
&\quad\times f(t,z(t))\,\mathrm{d}\theta\,\mathrm{d}t\Big]\,\mathrm{d}s.
\end{aligned}\tag{10}$$

According to (8), (9) and (10), we have

$$\begin{aligned}
\nu(\gamma)=\int_0^\infty e^{-\gamma t}\Big[&q\int_0^\infty\Phi_q(\theta)\mathcal{T}\left(\frac{t^q}{\theta^q}\right)\frac{t^{q-1}}{\theta^q}z_0\,\mathrm{d}\theta\\
&+h(t,z(t))+q\int_0^t\int_0^\infty\Phi_q(\theta)\mathcal{A}\mathcal{T}\left(\frac{(t-s)^q}{\theta^q}\right)\\
&\times h(s,z(s))\frac{(t-s)^q}{\theta^q}\,\mathrm{d}\theta\,\mathrm{d}s+q\int_0^t\int_0^\infty\Phi_q(\theta)\\
&\times\mathcal{T}(\frac{(t-s)^q}{\theta^q})\frac{(t-s)^q}{\theta^q}f(s,z(s))\,\mathrm{d}\theta\,\mathrm{d}s\Big]\,\mathrm{d}t.
\end{aligned}$$

Inverting the previous Laplace transform, we get

$$\begin{aligned}
z(t)=\;&q\int_0^\infty\Phi_q(\theta)\mathcal{T}\left(\frac{t^q}{\theta^q}\right)\frac{t^{q-1}}{\theta^q}z_0\,\mathrm{d}\theta+h\left(t,z(t)\right)\\
&+q\int_0^t\int_0^\infty\Phi_q\left(\theta\right)\mathcal{A}\mathcal{T}\left(\frac{(t-s)^q}{\theta^q}\right)\\
&\times h\left(s,z(s)\right)\frac{(t-s)^q}{\theta^q}\,\mathrm{d}\theta\,\mathrm{d}s\\
&+q\int_0^t\int_0^\infty\Phi_q\left(\theta\right)\mathcal{T}\left(\frac{(t-s)^q}{\theta^q}\right)\frac{(t-s)^q}{\theta^q}\\
&\times f\left(s,z(s)\right))\,\mathrm{d}\theta\,\mathrm{d}s\\
=\;&q\int_0^\infty\frac{1}{q}\theta^{-1-\frac{1}{q}}\Phi_q\left(\theta^{-\frac{1}{q}}\right)\theta\mathcal{T}\left(t^q\theta\right)t^{q-1}z_0\,\mathrm{d}\theta\\
&+h\left(t,z(t)\right)\\
&+q\int_0^t\int_0^\infty\theta\left(t-s\right)^{q-1}\frac{1}{q}\theta^{-1-\frac{1}{q}}\Phi_q\left(\theta^{-\frac{1}{q}}\right)\\
&\times\mathcal{A}\mathcal{T}\left(\left(t-s^q\right)\theta\right)h\left(s,z(s)\right)\,\mathrm{d}\theta\,\mathrm{d}s\\
&+q\int_0^t\int_0^\infty\theta\left(t-s\right)^{q-1}\frac{1}{q}\theta^{-1-\frac{1}{q}}\Phi_q\left(\theta^{-\frac{1}{q}}\right)\\
&\times\mathcal{T}\left(\left(t-s^q\right)\theta\right)f\left(s,z(s)\right)\,\mathrm{d}\theta\,\mathrm{d}s.
\end{aligned}$$

Set

$$\Psi_q\left(\theta\right)=\frac{1}{q}\theta^{-1-\frac{1}{q}}\Phi_q\left(\theta^{-\frac{1}{q}}\right)$$

and

$$\mathcal{T}_q(t)z=q\int_0^\infty\theta\Psi_q(\theta)\mathcal{T}(t^q\theta)z\,\mathrm{d}\theta.$$

The proof is complete. ■

The following lemma will be used throughout this paper.

Lemma 3. (Zhou and He, 2021) *The operator $\mathcal{T}_q(t)$ has the following properties:*

(i) For a fixed $t\geq 0$,

$$\|\mathcal{T}_q(t)z\|\leq\frac{M}{\Gamma(q)}\|z\|,\quad M>0,\quad\textit{for all } z\in Z,$$

and $\mathcal{T}_q(t)$ is linear and bounded.

(ii) $(\mathcal{T}_q(t))_{t\geq 0}$ is strongly continuous.

(iii) For any $z\in Z$, $\alpha\in]0,1[$, we have

$$\|A^\alpha\mathcal{T}_q(t)z\|\leq\frac{\alpha M_\alpha\Gamma(2-\alpha)}{t^{q\alpha}\Gamma(1+q(1-\alpha))}\|z\|.$$

Lemma 4. *Let $-\mathcal{A}$ generate a differentiable semigroup $\mathcal{T}(t)$. Then, for $z\in Z$, we have*

$$\mathcal{T}_q(t)z\in D(\mathcal{A}),\quad\forall t>0,$$

$$\mathcal{T}_q(t)\mathcal{T}_q(s)=\mathcal{T}_q(s)\mathcal{T}_q(t),\quad\forall t,s\geq 0,$$

and

$$\frac{\mathrm{d}\mathcal{T}_q^2(t)z}{\mathrm{d}t}=2\mathcal{T}_q(t)\frac{\mathrm{d}\mathcal{T}_q(t)z}{\mathrm{d}t},\quad\forall t>0.$$

Proof. For $t,s\geq 0$, we have

$$\begin{aligned}
\mathcal{T}_q(t)\mathcal{T}_q(s)&=q\int_0^\infty\theta\Psi_q(\theta)\mathcal{T}(t^q\theta)\,\mathrm{d}\theta\\
&\quad\times q\int_0^\infty\theta\Psi_q(\theta)\mathcal{T}(s^q\theta)\,\mathrm{d}\theta\\
&=q\int_0^\infty\theta\Psi_q(\theta)\mathcal{T}(s^q\theta)\,\mathrm{d}\theta\\
&\quad\times q\int_0^\infty\theta\Psi_q(\theta)\mathcal{T}(t^q\theta)\,\mathrm{d}\theta\\
&=\mathcal{T}_q(s)\mathcal{T}_q(t)\cdot
\end{aligned}$$

For any $z \in Z$ and $t > 0$, we get

$$\begin{aligned}\frac{\mathrm{d}\mathcal{T}_q(t)^2}{\mathrm{d}t} z &= \frac{\mathrm{d}}{\mathrm{d}t}\Big[q\int_0^\infty \theta\Psi_q(\theta)\mathcal{T}(t^q\theta)\,\mathrm{d}\theta z\Big]^2 \\ &= 2q\int_0^\infty \theta\Psi_q(\theta)\mathcal{T}(t^q\theta)\,\mathrm{d}\theta \\ &\quad\times \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^\infty \theta\Psi_q(\theta)\mathcal{T}(t^q\theta)\,\mathrm{d}\theta\right] z \\ &= 2\mathcal{T}_q(t)\frac{d\mathcal{T}_q(t)}{\mathrm{d}t} z.\end{aligned}$$

■

3.2. Uniqueness of the mild solution. In this subsection, our main purpose is to give sufficient conditions for the uniqueness of the mild solution to the system (1). Motivated by Lemma 2, we shall define the mild solution of the problem (1).

Definition 2. A function $z \in C_{1-q}(J, Z)$ is called a *mild solution* of the problem (1) if it satisfies the following fractional integral equation:

$$\begin{aligned} z(t) &= t^{q-1}\mathcal{T}_q(t)z_0 + h\left(t, z(t)\right) \\ &\quad - \int_0^t (t-s)^{q-1}\,\mathcal{A}\mathcal{T}_q\left(t-s\right)h\left(s, z(s)\right)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{q-1}\,\mathcal{T}_q\left(t-s\right)Bu(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{q-1}\,\mathcal{T}_q\left(t-s\right)f\left(s, z(s)\right)\,\mathrm{d}s. \end{aligned} \tag{11}$$

Before proving the existence and uniqueness of the *mild solution* of the system (1), we first formulate the following hypotheses:

H_0: $\mathcal{T}(t)$ is a compact operator for every $t > 0$.

H_1: There are $A_1 > 0$, $A_2 > 0$, and $\beta \in]0,1[$ such that $h \in D(\mathcal{A}^\beta)$, and the function $\mathcal{A}^\beta h(t,\cdot)$ satisfies

$$\begin{aligned}\left|\mathcal{A}^\beta h(t,y) - \mathcal{A}^\beta h(t,z)\right| &\le A_1 t^{1-q}\|y-z\| \\ \forall\ y,z \in PW_q,&\quad t\in[0,T],\end{aligned} \tag{12}$$

and

$$\left|\mathcal{A}^\beta h(t,z)\right| \le A_2 t^{1-q}\|z\|. \tag{13}$$

H_2: There is $c > 0$ and $\phi(\cdot) \in L^p(J,\mathbb{R}^+)$, $p > 1/q \ge 1$ such that

$$|f(t,z)\| \le \phi(t) + ct^{1-q}\|z\| \tag{14}$$

for a.e. $t \in J$ and all $z \in Z^q$.

H_3: There is $M_f > 0$ such that

$$\begin{aligned}\|f(t,y) - f(t,z)\| &\le M_f t^{1-q}\|y-z\| \\ &\forall y,z \in Z^q,\quad t\in J.\end{aligned} \tag{15}$$

Remark 1.

(i) Assumption H_0 is needed to prove that the set

$$\begin{aligned}V(t) = \Big\{q\int_0^\infty \theta\Psi_q(\theta)\mathcal{T}(t^q\theta)zd\theta,& \\ z\in Z,\quad \|z\|\le k,\ k>0\Big\}&\end{aligned}$$

is relatively compact in Z, which will be used to prove the compactness of the operator $\mathcal{T}_q(t)$.

(ii) Assumption H_1 is necessary to have the continuity of the operator

$$(t-s)^{q-1}\mathcal{A}\mathcal{T}_q(t-s)h(s,z(s)),\quad s\in[0,t[,$$

which will be used to obtain the existence and uniqueness of the mild solution and the approximate controllability of the system (1).

Theorem 1. *Assume that H_0–H_3 are satisfied. Furthermore, if*

$$\left[T^{1-q}|\mathcal{A}^{-\beta}|A_1 + \frac{\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}M_{1-\beta}A_1T^{q(\beta-1)+1} + TM_f\frac{M}{q\Gamma(q)}\right] < 1, \tag{16}$$

then, for each control function $u(\cdot) \in U$, the system (1) *has a unique mild solution on $C_{1-q}(J,Z)$.*

Proof. For each positive constant k, let $B_k = \{z \in C_{1-q}(J,Z) : \|z\|_{C_{1-q}} \le k\}$. Obviously, B_k is a bounded, convex and closed subset of $C_{1-q}(J,Z)$. We prove that the operator $\mathcal{L}$ defined by

$$\begin{aligned}(\mathcal{L}z)(t) &= t^{q-1}\mathcal{T}_q(t)z_0 + h(t,z(t)) \\ &\quad - \int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{T}_q(t-s)h(s,z(s))\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)Bu(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f(s,z(s))\,\mathrm{d}s,\end{aligned}$$

has a fixed point $z(\cdot)$ in B_k for any $u(\cdot) \in U$. Then $z(\cdot)$ is a mild solution of the system (1).

We first show that the operator $\mathcal{L}$ maps B_k into itself. For any $z \in B_k$, $t \in [0,T]$, we have, by H_0–H_3 and Lemma (3), that

$$\begin{aligned}&t^{1-q}\|(\mathcal{L}z(t))\| \\ &\quad\le \|\mathcal{T}_q(t)z_0\| + t^{1-q}\|h(t,z(t))\| \\ &\qquad + t^{1-q}\int_0^t (t-s)^{q-1}\|\mathcal{A}\mathcal{T}_q(t-s)h(s,z(s)\|\,\mathrm{d}s \\ &\qquad + t^{1-q}\int_0^t (t-s)^{q-1}\|\mathcal{T}_q(t-s)Bu(s)\|\,\mathrm{d}s \\ &\qquad + t^{1-q}\int_0^t (t-s)^{q-1}\|\mathcal{T}_q(t-s)f(s,z(s)\|\,\mathrm{d}s.\end{aligned}$$

Following the ideas of Zhou and Jiao (2010a), we obtain

$$\begin{aligned}&t^{1-q}\|(\mathcal{L}z(t))\|\\&\le \frac{M}{\Gamma(q)}\|z_0\|+\frac{M}{\Gamma(q)}\Big(\frac{p-1}{qp-1}\Big)^{\frac{p-1}{p}}T^{1-\frac{1}{p}}\|\Phi\|_{L^p}\\&+\frac{M}{\Gamma(q)}\Big(\frac{p-1}{qp-1}\Big)^{\frac{p-1}{p}}T^{1-\frac{1}{p}}\|Bu\|_{L^p}\\&+\Bigg[A_2|\mathcal{A}^{-\beta}|T^{1-q}+\frac{Tc}{q}\\&+\frac{\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}M_{1-\beta}A_2T^{q(\beta-1)+1}\Bigg]k.\end{aligned}$$

For a sufficiently large $k>0$, we have

$$t^{1-q}\|\mathcal{L}z\|\le k.$$

Therefore, we obtain

$$\|\mathcal{L}z\|_{C_{1-q}}\le k.$$

Then $\mathcal{L}(B_k)\subset B_k$.

Next, we will prove that the operator $\mathcal{L}$ is a contraction on B_k, for every $y,z\in Z$ and $t\in[0,T]$. For every $z,y\in B_k$ and $t\in[0,T]$, we can deduce, by H_0–H_3 and Lemma (3), that

$$\begin{aligned}&\|(\mathcal{L}z(t))-(\mathcal{L}y(t))\|\\&\le t^{1-q}\|h(t,z(t))-h(t,y(t))\|\\&+t^{1-q}\int_0^t(t-s)^{q-1}\|\mathcal{A}\mathcal{T}_q(t-s)\\&\times\Big(h(s,z(s))-h(s,y(s))\Big)\|\,\mathrm{d}s\\&+t^{1-q}\int_0^t(t-s)^{q-1}\mathcal{T}_q(t-s)\\&\times\Big(f(s,z(s))-f(s,y(s))\Big)\|\,\mathrm{d}s\\&\le\Bigg[T^{1-q}|\mathcal{A}^{-\beta}|A_1+\frac{\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}M_{1-\beta}\\&A_1T^{q(\beta-1)+1}+TM_f\frac{M}{q\Gamma(q)}\Bigg]\|z-y\|_{C_{1-q}}.\end{aligned}$$

Consequently, we deduce by (16) that the operator $\mathcal{L}$ is a contraction on B_k. According to the Banach fixed point theorem, $\mathcal{L}$ has a unique fixed point in $C_{1-q}(J,Z)$. The proof is completed. ■

4. Approximate controllability results for a neutral system

We assume that the system (1) has a mild solution for $z_0\in Z$ and a control $u(\cdot)\in\mathcal{U}$, which will be denoted by $z(t;0,z_0,u)$. Let $U_d=\{z(T;0,z_0,u):u(\cdot)\in\mathcal{U}\}$ be the reachable set of system (1) at terminal time T.

Definition 3. The system (1) is said to be *approximately controllable* on J if $\overline{U_d}=Z$, for all $z_0\in Z$.

Consider the Nemytskii operator related to f defined by

$$k_f:\begin{cases}C_{1-q}(J,Z)\to L^p(J,Z),\\ z\to f(t,z(t)).\end{cases}$$

Define the operator $\mathcal{K}:L^p(J,Z)\to Z$,

$$\mathcal{K}h=\int_0^b(b-s)^{q-1}\mathcal{T}_q(b-s)h(s)\,\mathrm{d}s,\quad h(\cdot)\in L^p(J,Z),$$

which is linear and bounded. In what follows, we assume that the hypothesis below is true:

H_4: For $\varepsilon>0$ and $\varphi(\cdot)\in L^p(J,Z)$, there is a $u(\cdot)\in L^p(J,U)$ such that

$$\|\mathcal{K}\varphi-\mathcal{K}\mathcal{B}u\|<\varepsilon,\tag{17}$$

$$\|\mathcal{B}u(\cdot)\|_{L^p}\le L\|\varphi(\cdot)\|_{L^p},\tag{18}$$

where L is a constant which is independent of $\varphi(\cdot)\in L^p(J,Z)$, and

$$\begin{aligned}L\mathcal{K}'E_{q\beta}\Bigg(\frac{T^{1-q(1-\beta)}}{\alpha'}\Bigg[\frac{\Gamma(\beta)}{\Gamma(q\beta)}A_1M_{1-\beta}\\+\frac{MM_f}{\Gamma(q)}\Bigg]\Gamma(q\beta)\Bigg)<1,\end{aligned}\tag{19}$$

while E_β is the Mittag-Leffler function given by

$$E_\beta(y)=\sum_{k=0}^{k=\infty}\frac{y^k}{\Gamma(k\beta+1)}.$$

The main result of this section requires the following proposition.

Proposition 1. *Under Assumptions H_1–H_3 on the nonlinearity of f, if $\alpha=1-T^{1-q}\left|\mathcal{A}^{-\beta}\right|A_2>0$, $\alpha'=1-T^{1-q}\left|\mathcal{A}^{-\beta}\right|A_1>0$, then any mild solution of the system* (1) *fulfills*

$$\begin{aligned}&\|z(\cdot;0,z_0,u)\|_{C_{1-q}}\\&\le\mathcal{K}E_{q\beta}\Bigg(\frac{M}{\alpha\Gamma(q)}T^{1-q(1-\beta)}\\&\times\Big[c+\frac{\Gamma(\beta)}{\Gamma(q\beta)}M_{1-\beta}A_2\Big]\Gamma(q\beta)\Bigg),\end{aligned}$$

for any $u(\cdot)\in\mathcal{U}$, and

$$\begin{aligned}&\|z_1(\cdot)-z_2(\cdot)\|_{C_{1-q}}\\&\le\mathcal{K}'E_{q\beta}\Bigg(\frac{T^{1-q(1-\beta)}}{\alpha'}\Bigg[\frac{\Gamma(\beta)}{\Gamma(q\beta)}A_1M_{1-\beta}\\&+\frac{MM_f}{\Gamma(q)}\Bigg]\Gamma(q\beta)\Bigg)\|\mathcal{B}u_1(\cdot)-\mathcal{B}u_2(\cdot)\|_{L^p},\end{aligned}$$

for any $u_1,\ u_2 \in \mathcal{U}$, where

$$\mathcal{K} = \frac{M}{\alpha\Gamma(q)}\Big[\|z_0\| + \Big(\frac{p-1}{qp-1}\Big)^{\frac{p-1}{p}} \times T^{1-\frac{1}{p}}\Big(\|\mathcal{B}u\|_{L^p} + \|\Phi\|_{L^p}\Big),$$

$$\mathcal{K}' = \frac{M}{\alpha'\Gamma(q)}T^{1-q(1-\beta)}\Big[c + \frac{\Gamma(\beta)}{\Gamma(q\beta)}M_{1-\beta}A_2\Big]\Gamma(q\beta).$$

Proof. Let z be a mild solution of (1), with respect to $u(\cdot) \in \mathcal{U}$ on $C_{1-q}(J,Z)$. We have

$$\begin{aligned} z(t) &= t^{1-q}\mathcal{T}_q(t)z_0 + h(t,z(t)) \\ &\quad - \int_0^t (t-s)^{q-1}\mathcal{A}\mathcal{T}_q(t-s)h(s,z(s))\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)\mathcal{B}u(s)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f(s,z(s))\,\mathrm{d}s. \end{aligned}$$

For $t \in J$, we can get

$$\begin{aligned} &t^{1-q}\|z(t)\| \\ &\le \|\mathcal{T}_q(t)z_0\| + t^{1-q}\|\mathcal{A}^{-\beta}\mathcal{A}^{\beta}h(t,z(t))\| \\ &\quad + t^{1-q}\int_0^t (t-s)^{q-1}\|\mathcal{A}^{1-\beta}\mathcal{T}_q(t-s) \\ &\quad \times \mathcal{A}^{\beta}h(s,z(s))\|h(s,z(s))\,\mathrm{d}s \\ &\quad + t^{1-q}\int_0^t (t-s)^{q-1}\|\mathcal{T}_q(t-s)f(s,z(s))\|\,\mathrm{d}s \\ &\quad + t^{1-q}\int_0^t (t-s)^{q-1}\|\mathcal{T}_q(t-s)\mathcal{B}u(s)\|\,\mathrm{d}s \\ &\le \frac{M}{\Gamma(q)}\|y_0\| + A_2T^{1-q}|\mathcal{A}^{-\beta}|t^{1-q}\|z(t)\| \\ &\quad + t^{1-q}\frac{q\Gamma(1+\beta)}{\Gamma(1+q\beta)}M_{1-\beta}A_2 \\ &\quad \times \int_0^t (t-s)^{q\beta-1}s^{1-q}\|z(s)\|\,\mathrm{d}s \\ &\quad + \frac{M}{\Gamma(q)}t^{1-q}\int_0^t (t-s)^{q-1}\|Bu(s)\|_{L^p}\,\mathrm{d}s \\ &\quad + \frac{M}{\Gamma(q)}t^{1-q} \\ &\quad \times \int_0^t (t-s)^{q-1}\Big[\Phi(s) + cs^{1-q}\|z(s)\|\Big]\,\mathrm{d}s. \end{aligned} \tag{20}$$

Then

$$\begin{aligned} &\Big[1 - T^{1-q}|\mathcal{A}^{-\beta}|A_2\Big]t^{1-q}\|z(t)\| \\ &\le \frac{M}{\Gamma(q)}\Bigg[\|z_0\| \\ &\quad + \Big(\frac{p-1}{qp-1}\Big)^{\frac{p-1}{p}}T^{1-\frac{1}{p}}\Big(\|\mathcal{B}u\|_{L^p} + \|\Phi\|_{L^p}\Big)\Bigg] \\ &\quad + \frac{Mt^{1-q}}{\Gamma(q)}\Big[c + \frac{q\Gamma(1+\beta)}{\Gamma(1+q\beta)}M_{1-\beta}A_2\Big] \\ &\quad \times \int_0^t (t-s)^{q\beta-1}s^{1-q}\|z(s)\|\,\mathrm{d}s. \end{aligned} \tag{21}$$

Set $\alpha = 1 - T^{1-q}|\mathcal{A}^{-\beta}|A_2$, which implies that

$$\begin{aligned} &t^{1-q}\|z(t)\| \\ &\le \frac{M}{\alpha\Gamma(q)}\Bigg[\|z_0\| + \Big(\frac{p-1}{qp-1}\Big)^{\frac{p-1}{p}}T^{1-\frac{1}{p}} \\ &\quad \times \Big(\|\mathcal{B}u\|_{L^p} + \|\Phi\|_{L^p}\Big)\Bigg] \\ &\quad + \frac{M}{\alpha\Gamma(q)}T^{1-q}\Bigg[c + \frac{q\Gamma(1+\beta)}{\Gamma(1+q\beta)}M_{1-\beta}A_2\Bigg] \\ &\quad \times \int_0^t (t-s)^{q\beta-1}s^{1-q}\|y(s)\|\,\mathrm{d}s. \end{aligned} \tag{22}$$

Let

$$P(t) = t^{1-q}\|z(t)\|.$$

Then

$$\begin{aligned} P(t) &\le \mathcal{K} + \frac{M}{\alpha\Gamma(q)}T^{1-q}\Bigg[c + \frac{q\Gamma(1+\beta)}{\Gamma(1+q\beta)}M_{1-\beta}A_2\Bigg] \\ &\quad \times \int_0^t (t-s)^{q\beta-1}P(s)\,\mathrm{d}s \\ &\le \mathcal{K} + \frac{M}{\alpha\Gamma(q)}T^{1-q}\Big[c + \frac{\Gamma(\beta)}{\Gamma(q\beta)}M_{1-\beta}A_2\Big] \\ &\quad \times \int_0^t (t-s)^{q\beta-1}P(s)ds. \end{aligned}$$

From Corollary 2 of Ye *et al.* (2007), we obtain

$$P(t) \le \mathcal{K}E_{q\beta}\Bigg(\frac{M}{\alpha\Gamma(q)}b^{1-q} \times \Big[c + \frac{\Gamma(\beta)}{\Gamma(q\beta)}M_{1-\beta}A_2\Big]T^{q\beta}\Gamma(q\beta)\Bigg).$$

Therefore,

$$\begin{aligned}\|z(t)\|_{C_{1-q}} &= \sup_{t\in J} t^{1-q}\|z(t)\| \\ &\le \mathcal{K}E_{q\beta}\left(\frac{M}{\alpha\Gamma(q)}T^{1-q(1-\beta)}\Big[c \right. \\ &\quad \left. + \frac{\Gamma(\beta)}{\Gamma(q\beta)}M_{1-\beta}A_2\Big]\Gamma(q\beta)\right).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&\|z(t)-y(t)\|_{C_{1-q}} \\ &\le \mathcal{K}'E_{q\beta}\left(\frac{1}{\alpha'}\Big[T^{1-q(1-\beta)}\frac{\Gamma(\beta)}{\Gamma(q\beta)}A_1M_{1-\beta}\right. \\ &\quad \left. + t^{1-q}\frac{MM_f}{\Gamma(q)}\Big]\Gamma(q\beta)\right)\|\mathcal{B}u_z(\cdot)-\mathcal{B}u_y(\cdot)\|_{L^p},\end{aligned}$$

which completes the proof. ■

Now, we are in the position to prove the approximate controllability of the system (1).

Theorem 2. *If* H_2, H_3 *and* H_4 *are fulfilled, then the system* (1) *is approximately controllable on* J *if* $\mathcal{A}$ *generates a differentiable semigroup* $\mathcal{T}(t)$ *on the Banach space* Z.

Proof. Since $\overline{D(\mathcal{A})} = Z$, it is enough to show that $D(\mathcal{A}) \subset U_d$, i.e., $\forall\varepsilon > 0$ and $z_d \in D(\mathcal{A}), \exists u_\varepsilon \in \mathcal{U}$ such that

$$\begin{aligned}&\|z_d - T^{q-1}\mathcal{T}_q(T)z_0 - h(T,z(T)) \\ &\quad - A\mathcal{K}k_h(z_\varepsilon) - \mathcal{K}k_f(z_\varepsilon) - \mathcal{K}\mathcal{B}u_\varepsilon\| \le \varepsilon,\end{aligned}$$

where $z_\varepsilon(t) = z(t;0,z_0,u_\varepsilon)$ and $t\in[0,T]$.

Since $\mathcal{T}(t)$ is differentiable, for any $z_0 \in Z$, we have $T^{q-1}\mathcal{T}_q(T)z_0 \in D(\mathcal{A})$. Therefore, for any $z_d \in D(\mathcal{A})$, there exists $\xi(\cdot) \in L^p(J,Z)$ such that $\mathcal{K}\xi = z_d - T^{q-1}\mathcal{T}_q(T)z_0 - h(T,z_T) - A\mathcal{K}k_h(z_\varepsilon)$. For example,

$$\begin{aligned}\xi(t) &= \frac{[\Gamma(q)]^2(T-t)^{1-q}}{T}\left[\mathcal{T}_q(T-t) - 2t\frac{\mathrm{d}\mathcal{T}_q(T-t)}{\mathrm{d}t}\right] \\ &\quad \times \left[z_d - T^{q-1}\mathcal{T}_q(T)z_0 - h(T,z(T))\right] \\ &\quad - Ak_h(z_\varepsilon), \quad t\in]0,T[.\end{aligned}$$

From H_4, for any $\varepsilon > 0$ and $u_1(\cdot)\in\mathcal{U}$, there is a $u_2(\cdot)\in\mathcal{U}$ such that

$$\|\mathcal{K}\xi - \mathcal{K}\Phi_f(x_1) - \mathcal{K}\mathcal{B}u_2\| \le \frac{\varepsilon}{2^2},$$

where $z_1(t) = z(t;0,z_0,u_1)$, $t\in[0,T]$. Write $z_2(t) = z(t;0,z_0,u_2)$, $t\in[0,T]$ by H_4, again there exists $u'(\cdot)\in\mathcal{U}$ such that

$$\|\mathcal{K}[k_f(z_2) - k_f(z_1)] - \mathcal{K}\mathcal{B}u'\|_Z \le \frac{\varepsilon}{2^2}$$

and

$$\begin{aligned}&\|\mathcal{B}u'(\cdot)\|_{L^p} \\ &\le L\|k_f(z_2)(\cdot) - k_f(z_1)(\cdot)\| \\ &\le LM_f t^{1-q}\|z_2(\cdot) - z_1(\cdot)\| \\ &\le \mathcal{K}'E_{q\beta}\left(\frac{1}{\alpha'}\Big[T^{1-q}\frac{\Gamma(\beta)}{\Gamma(q\beta)}A_1M_{1-\beta}\right. \\ &\quad \left. + t^{1-q}\frac{MM_f}{\Gamma(q)}\Big]\Gamma(q\beta)T^{q\beta}\right)\|\mathcal{B}u_2(\cdot) - \mathcal{B}u_1(\cdot)\|_{L^p}.\end{aligned}$$

Then define

$$u_3(t) = u_2(t) - u'(t), \quad u_3(\cdot)\in\mathcal{U}.$$

Accordingly, we obtain

$$\begin{aligned}&\|\mathcal{K}\xi - \mathcal{K}k_f(z_2) - \mathcal{K}\mathcal{B}u_3\| \\ &\le \|\mathcal{K}\xi - \mathcal{K}k_f(z_1) - \mathcal{K}\mathcal{B}u_2\| \\ &\quad + \|\mathcal{K}\mathcal{B}u' - [\mathcal{K}k_f(z_2) - \mathcal{K}k_f(z_1)]\| \\ &\le \left(\frac{1}{2^2} + \frac{1}{2^3}\right)\varepsilon.\end{aligned}$$

By induction, we get

$$\|\mathcal{K}\xi - \mathcal{K}k_f(z_n) - \mathcal{K}\mathcal{B}u_{n+1}\| \le \left(\frac{1}{2^2} + \dots + \frac{1}{2^n}\right)\varepsilon,$$

where $z_n = z(\cdot,0,z_0,u_n)$, $t\in[0,T]$, and

$$\begin{aligned}&\|\mathcal{B}u_{n+1}(\cdot) - \mathcal{B}u_n(\cdot)\|_{L^p} \\ &\le L\mathcal{K}'E_{q\beta}\left(\frac{T^{1-q(1-\beta)}}{\alpha A'}\left[\frac{\Gamma(\beta)}{\Gamma(q\beta)}A_1M_{1-\beta}\right.\right. \\ &\quad \left.\left. + \frac{MM_f}{\Gamma(q)}\right]\Gamma(q\beta)\right)\|\mathcal{B}u_n(\cdot) - \mathcal{B}u_{n-1}(\cdot)\|_{L^p}.\end{aligned}$$

It follows from (19) that $\{\mathcal{B}u_n(\cdot)\}_{n\in\mathbb{N}^\star}$ is a Cauchy sequence on $L^p(J,Z)$; then

$$\begin{aligned}&\exists\, w(\cdot)\in L^p(J,Z) \\ &\text{such that } \mathcal{B}u_n(\cdot)\to w(\cdot) \text{ in } L^p(J,Z), \text{ as } n\to+\infty.\end{aligned}$$

Thus, for all $\varepsilon > 0$, there exists $N\in\mathbb{N}^\star$ such that $\|\mathcal{K}\mathcal{B}u_{N+1} - \mathcal{K}\mathcal{B}u_N\| \le \varepsilon/2\cdot$

Furthermore,

$$\begin{aligned}&\|\mathcal{K}\xi - \mathcal{K}k_f(z_N) - \mathcal{K}\mathcal{B}u_N\| \\ &\le \|\mathcal{K}\xi - \mathcal{K}k_f(z_N) - \mathcal{K}\mathcal{B}u_{N+1}\| \\ &\quad + \|\mathcal{K}\mathcal{B}u_{N+1} - \mathcal{K}\mathcal{B}u_N\| \\ &\le \left(\frac{1}{2^2} + \dots + \frac{1}{2^n}\right)\varepsilon + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Therefore, the system (1) is approximately controllable on J. ■

Remark 2. If $h(t,z(t)) \equiv 0$, $\forall t \in (0,T]$, then the system (1) is transformed to the fractional system given by

$$\begin{cases} D^q_{RL}z(t) = -\mathcal{A}z(t) + \mathcal{B}u(t) + f(t,z(t)) \\ \quad t \in J = (0,T] \quad (T \geq 1), \\ I^{1-q}_t z(t)|_{t=0} = z_0. \end{cases}$$

which is, according to Theorem 4.3 by Liu and Li (2015), approximately controllable under Assumptions H_2, H_3 and H_4.

5. Application

Consider the following system, and apply the previous results to it:

$$\begin{cases} D^{2/3}_{RL}\Big(w(t,x) - \int_0^\pi G(x,y)w(t,y)\,\mathrm{d}y\Big) = \Delta w(t,x) \\ +\rho t^{\frac{2}{3}}\sin(w(t,x)) + \mathcal{B}u(t,x) \;\; t \in [0,1], \;\; x \in \Omega, \\ w(t,0) = w(t,\pi) = 0, \;\; t \in [0,1], \\ I^{\frac{1}{3}}_{0^+}\left[w(t,x) - \int_0^\pi G(x,y)w(t,y)\,\mathrm{d}y\right]|_{t=0} = w_0, \end{cases} \tag{23}$$

where $\Omega = [0,\pi]$, $Z = L^2([0,\pi],\mathbb{R})$, $U = L^2([0,\pi])$ and $\rho \geq 0$, the operator $\mathcal{A}$ is given by $\mathcal{A}z = -z''$, and its domain is

$$D(\mathcal{A}) := \Big\{z \in Z \;:\; z, z' \text{ are absolutely continuous}, \\ z'' \in Z, \quad z(0) = z(\pi) = 0\Big\}.$$

The operator $-\mathcal{A}$ generates a uniformly bounded differentiable strongly continuous semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ which is compact, analytic and self-adjoint. Then, let $\gamma_n = n^2$ and

$$\psi_n(y) = \sqrt{\frac{2}{\pi}}\sin(ny)$$

be, respectively, the eigenvalues and eigenfunctions of $\mathcal{A}$, for all $n \in \mathbb{N}^\star$. Thus, $0 < \gamma_1 \leq \gamma_2 \leq \cdots, \gamma_n \to \infty$ as $n \to \infty$, and $\{\psi_n\}_{n=1}^\infty$ form an orthonormal basis of Z. Moreover,

$$\mathcal{A}z = \sum_{n=1}^\infty \gamma_n \langle x, \psi_n\rangle \psi_n, \quad z \in D(\mathcal{A}).$$

We also have the following:

(i) $\mathcal{T}(t)z = \sum_{n=1}^{n=\infty} e^{-n^2 t}\langle x, \psi_n\rangle\psi_n$. In particular, $\|\mathcal{T}(t)\| \leq e^{-t} < 1 = M$.

(ii) For all $z \in Z$, $\mathcal{A}^{\frac{-1}{2}}z = \sum_{n=1}^{n=\infty}\frac{1}{n}\langle x,\psi_n\rangle\psi_n$, and $\|\mathcal{A}^{\frac{-1}{2}}\|_{L^2[0,\pi]} = 1$.

(iii) For all $z \in Z$,

$$\mathcal{A}^{\frac{1}{2}}z = \sum_{n=1}^{n=\infty} n\langle x,\psi_n\rangle\psi_n,$$

with

$$D\left(\mathcal{A}^{\frac{1}{2}}\right) = \left\{z(\cdot) \in Z \;:\; \sum_{n=1}^{n=\infty} n\langle x,\psi_n\rangle\psi_n \in Z\right\}.$$

Clearly, (2) and (3) are satisfied.

The system (23) can be formulated as the following control system in Z:

$$\begin{cases} {}^{RL}_{0}D^q_t\left[z(t) - h\left(t, z(t)\right)\right] \\ = -\mathcal{A}z(t) + \mathcal{B}u(t) + f(t,z(t)), \\ \qquad\qquad t \in J = (0,T], \; (T \geq 1), \\ I^{1-q}_t[z(t) - h(t,z_t)]|_{t=0} = z_0, \end{cases} \tag{24}$$

where $(z(t))(x) = w(t,x)$, $t \in [0,1]$, $x \in [0,\pi]$. The function $h : [0,1] \times C([0,1],Z) \to Z$ is given by

$$(h(t,z(t))(x) = \int_0^\pi G(x,y)w(t,y)\,\mathrm{d}y$$

Let

$$\left(G_h\left(v\right)\right)(s) = \int_0^\pi G\left(y,s\right)v\left(y\right)\,\mathrm{d}y,$$

for $v \in L^2\left([0,\pi],\mathbb{R}\right)$, $s \in [0,\pi]$.

In addition, assume that the following conditions hold:

(i) $(y,s) \to G(y,s)$ is a measurable function for all $y, s \in [0,\pi]$, and

$$\int_0^\pi\int_0^\pi G^2(y,s)\,\mathrm{d}y\,\mathrm{d}s < \infty,$$

(ii) $\partial G(y,s)$ is measurable, $G(0,y) = G(\pi,y) = 0$, and

$$\mathcal{A} = \Big(\int_0^\pi\int_0^\pi (\partial_z G(y,s))^2\,\mathrm{d}y\,\mathrm{d}s\Big)^{\frac{1}{2}} < \infty.$$

From (1), we have that G_h is a bounded and linear on $L^2([0,\pi],\mathbb{R})$ and $G_h(v) \in D\left((-\mathcal{A})^{\frac{1}{2}}\right)$, where $\|\mathcal{A}^{\frac{1}{2}}G_h\|_{L^2[0,\pi]} < \infty$. From the definition of G_h and (2), we obtain

$$\begin{aligned}\langle G_h v, \psi_n\rangle &= \int_0^\pi \psi_n(y)\Big(\int_0^\pi G(y,s)v(z)ds\Big)dy \\ &= \frac{1}{n}\sqrt{\frac{2}{\pi}}\langle \mathcal{G}(v), \cos(ny)\rangle,\end{aligned}$$

where

$$\left(\mathcal{G}\left(v\right)\right)(s) = \int_0^\pi \partial G\left(y,s\right)v\left(y\right)\,\mathrm{d}y.$$

From (2), we have that $\mathcal{G} : L^2([0,\pi],\mathbb{R}) \to L^2([0,\pi],\mathbb{R})$ is bounded and linear with $\|\mathcal{G}\|_{L^2[0,\pi]} \le \mathcal{A}$. Furthermore, $\|\mathcal{A}^{\frac{1}{2}} G_h(v)\|_{L^2[0,\pi]} = \|\mathcal{G}(v)\|_{L^2[0,\pi]}$; then (12) and (13) are satisfied.

The function $f : [0,1] \times C([0,1], Z) \to Z$ is given by

$$f(t,z) = \rho t^{\frac{2}{3}} \sin(z(t)).$$

For all $t \in [0,1]$, $\quad z \in [0,\pi]$, we have

$$|f(t,z)| \le t + \rho t^{1/3} |z(t)|.$$

Then the inequality (14) is satisfied.

Further, for any $z, y \in Z$, $t \in [0,1]$, we have

$$\begin{aligned} |f(t,z) - f(t,y)| &\le \rho t^{\frac{1}{3}} t^{\frac{1}{3}} |\sin(z(t)) - \sin(y(t))| \\ &\le \rho t^{\frac{1}{3}} |z(t) - y(t)|_{C_{\frac{1}{3}}([0,1],Z)}. \end{aligned}$$

Then, the inequality (15) holds.

Hence, according to Theorem (1), the system (23) has a unique mild solution provided that the inequality (16) of Theorem (1) is satisfied.

For $u(\cdot) \in \mathcal{V} = L^2([0,1], U)$, we get

$$u(t) = \sum_{n=1}^{n=\infty} u_n(t)\psi_n, \quad u_n(t) = \langle u(t), \psi_n \rangle.$$

The control operator $\mathcal{B}$ is defined by

$$\mathcal{B}u(t) = \sum_{n=1}^{\infty} \overline{u}_n(t)\psi_n,$$

with

$$\overline{u}_n(t) = \begin{cases} 0, & 0 \le t < 1 - \frac{1}{n^2}, \\ u_n(t), & 1 - \frac{1}{n^2} \le t \le 1, \end{cases}$$

$n = 1, 2, \dots$. Since $\|\mathcal{B}u(\cdot)\| \le \|u(\cdot)\|$, we have that $\mathcal{B} \in L(\mathcal{V}, L^2([0,1], Z))$.

Now, write

$$k = \int_0^1 (1-s)^{-\frac{1}{3}} \mathcal{T}_{\frac{2}{3}}(1-s)h(s)\,\mathrm{d}s = \sum_{n=1}^{\infty} k_n \psi_n,$$

$$k_n = \langle\ k, \psi_n \rangle, \ \forall\ k(\cdot) \in L^2([0,1], Z).$$

Choose $\tilde{u}_n(t)$,

$$\tilde{u}_n(t) = \frac{2n^2}{1 - e^{-2}} k_n e^{-n^2(1-t)}, \quad 1 - \frac{1}{n^2} \le t \le 1$$

and

$$k_n = \int_{1-\frac{1}{n^2}}^1 \int_0^\infty (1-t)^{-1/3} \theta \Psi_{2/3}(\theta) \times e^{-n^2\theta(1-t)^{\frac{2}{3}}} \tilde{u}_n(t)\,\mathrm{d}\theta\,\mathrm{d}t.$$

Define

$$u(t) = \sum_{n=1}^{\infty} u_n(t)\psi_n,$$

where

$$u_n(t) = \begin{cases} 0, & 0 \le t < 1 - \frac{1}{n^2}, \\ \tilde{u}_n(t), & 1 - \frac{1}{n^2} \le t \le 1, \end{cases}$$

$n = 1, 2, \dots$. Moreover, for each function $h(\cdot) \in L^2([0,1], Z)$, there exists $u(\cdot) \in \mathcal{V}$ such that

$$\int_0^1 (1-s)^{-\frac{1}{3}} \mathcal{T}_{\frac{2}{3}} \mathcal{B}u(s)\,\mathrm{d}s = \int_0^1 (1-s)^{-\frac{1}{3}} \mathcal{T}_{\frac{2}{3}}(1-s)h(s)\,\mathrm{d}s.$$

It is clear that (17) is satisfied. Furthermore, we have

$$\begin{aligned} &\|\mathcal{B}u(\cdot)\|^2 \\ &= \sum_{n=1}^{\infty} \int_{1-\frac{1}{n^2}}^1 |\tilde{u}_n(t)|^2\,\mathrm{d}t \\ &= (1-e^{-2})^{-1} \sum_{n=1}^{\infty} 2n^2 k_n^2 \\ &= \frac{2}{3}(1-e^{-2})^{-1} \sum_{n=1}^{\infty} (1 - e^{-2n^2}) \int_0^1 |h_n(t)|^2\,\mathrm{d}t \\ &\le \frac{3}{2}(1-e^{-2})^{-1} |h(\cdot)|^2. \end{aligned}$$

Then (18) is fulfilled, and the system (23) is approximately controllable on $[0,1]$ if

$$\frac{3}{2}(1-e^{-2})^{-1} K' E_{\frac{1}{3}} \left(\frac{1}{3} \left[\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3})} \mathcal{A} + \frac{\rho}{\Gamma(\frac{1}{2})} \right] \frac{1}{3} \right) < 1.$$

6. Conclusion

In this paper, we established sufficient conditions for the existence and uniqueness of the mild solution, and approximate controllability for a class of nonlinear fractional evolution neutral systems in the sense of a Riemann–Liouville derivative in Banach spaces. More precisely, we obtained the existence of the mild solution by using the Laplace transform and semigroup theory. Furthermore, we proved the uniqueness of the solution using the Banach fixed-point theorem. A proof of approximate controllability was also established by constructing a Cauchy sequence, which is a weaker concept present in most application problems. An example was given to validate our findings.

Our work can be extended to the case of hyperbolic systems in the deterministic case, for a stochastic one, and for a class of fractional neutral systems with damping (Dhayal *et al.*, 2019; Du *et al.*, 2020; Almarri and

Elshenhab, 2022; Mabel Lizzy and Balachandran, 2018). Various topics of interest remain open, for instance, the case of approximate controllability of a semilinear neutral evolution equation with impulses, delay and nonlocal conditions (Agarwal *et al*., 2022; Leiva and Sundar, 2017).

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Received: 14 August 2023
Revised: 9 November 2023
Accepted: 17 November 2023